Maximum Likelihood Estimation of the Heston stochastic volatility model using asset and option prices

Francesco Zirilli
Dipartimento di Matematica G. Castelnuovo
Università di Roma La Sapienza
00185 Roma, Italy
Ph. n. +39 06 49913282
Fax n. +39 06 44701007
Email: f.zirilli@caspur.it

we consider the Heston stochastic volatility model [1]:

(1) $$dX = -\frac{V}{2}dt + \sqrt{V}dW^1 \qquad , \; t > 0$$

(2) $$dV = -\gamma(V-\theta) + k\sqrt{V}dW^2 , \; t > 0$$

with the initial conditions

(3) $$X(0) = 0$$

(4) $$V(0) = v_0$$

where

(5) $$X(t) = \ell n \frac{S(t)}{S(0)} - \mu_d t , \; t \geq 0$$

is the centered log return of the asset price $S(t)$, and $V(t) = \sigma^2(t)$, $t \geq 0$ is the variance of the stock prices.

The quantities dW^1, dW^2 are stochastic differentials of standard Wiener processes whose correlation coefficient ρ is assumed to be constant. The Heston model (1), (2), (3), (4) depends on the real parameters: $\gamma, \theta, k, v_0, \rho$. Due to their economic meaning these parameters satisfy some constraints, that is:

(6) $$\gamma \geq 0$$

(7) $$\theta \geq 0$$

(8) $k \geq 0$

(9) $v_0 \geq 0$

(10) $-1 \leq \rho \leq 1$

(11) $\frac{2\gamma\theta}{k} > 1$

The constraint (11) guarantees that the volatility $V(t)$ remains positive for $t > 0$ if $v_0 > 0$.
We consider the problem of estimating the parameters $(\gamma, \theta, k, v_0, \rho)$ starting from the knowledge of asset and option prices. That is let $t_0 < t_1 ... < t_N$ be the (known) observation times, we assume $t_0 = 0$ and for $i = 0,1,...,N$ let X_i, c_i be the observed centered log return of the asset price (remind that $X_0 = 0$ and the observed price of an European call option on the asset described by the Heston model with maturity time $t = T > t_N$ and strike price K. We assume that:

(12) $X_i = X(t_i)$

(13) $c_i = C(t_i, T, K, X_i, V(t_i)) + \varepsilon_i, i = 0,1,2,...,N$

where $C(t, T, K, X_i, V(t_i))$ is the price at time $t < T$ of the European call on the asset described by the Heston model with maturity time $t = T$ and strike price *K*, $E(V(t_i))$ is the estimated value of $V(t_i)$ and $\varepsilon_i, i = 0,1,...,N$ are known independent Gaussian variables.
Let $p(x, v, t | X_i c_i, i \text{ such that } t_i \leq t)$ be the transition probability density associated to the Heston model of having $X(t) = x$, $V(t) = v$ conditioned on the measures (12), (13) made up to time *t*. This transition probability density is the solution of the Kushner equation, that in this case reduces to the solution of the Fokker Planck equation associated to (1), (2) between the observation times with initial conditions that take care of (12), (13) (see Jazwinski [2]).
Let

(14) $F(\gamma, \theta, k, v_0, \rho) =$

$$\sum_{i=0}^{N} \ell n \int_{\mathbf{R}^+} p(X_i, v, t_i | X_i c_i \; i \text{ such that } t_i \leq t) dv \, ,$$

The estimation problem is formulated as follows:

$$(15) \quad \max_{(\gamma,\theta,k,v_0,\rho)} F(\gamma,\theta,k,v_0,\rho)$$

Subject to (6), (7), (8), (9), (10), (11).
We present an ad hoc very efficient way of evaluating the transition probability density $p(x,v,t|X_i,c_i \; i \text{ such that } t_i \le t)$ based on the FFT and a wavelet expansion and we solve the optimization problem (15) via a standard optimization procedure. Finally some numerical examples are presented.

References

[1] S. Heston: A closed form solution for options with stochastic volatility with applications to bonds and currency options, Review of Financial Studies 6, (1993), 327-343.

[2] A.H. Jazwinski: Stochastic processes and filtering theory, Academic Press, New York, 1970.