MONTE CARLO SIMULATION OF GENERALIZED GAUSSIAN DENSITIES

Martina Nardon[1], Paolo Pianca[1]

[1] Dipartimento di Matematica Applicata
Università Ca' Foscari di Venezia
(e-mail: mnardon@unive.it, pianca@unive.it)

ABSTRACT: This contribution deals with Monte Carlo simulation techniques for generalized Gaussian random variables. Such a parametric family of distributions has been proposed in many applications in science to describe physical phenomena and it seems interesting also in modeling economic and financial data. For low values of the shape parameter α, the distribution presents heavy tails. In particular, the choice of $\alpha = 1/2$ is considered for which a fast and accurate simulation procedure is analyzed.

KEYWORDS: Generalized Gaussian density, heavy tails, Lambert W function, Monte Carlo simulation.

1 The generalized Gaussian density

The parametric family of generalized Gaussian densities has been used to model successfully many phenomena in science as, for example, in the area of signal processing (see Kokkinakis & Nandi, 2005).

The probability density function (PDF) of a generalized Gaussian random variable X, with mean μ and variance σ^2, is defined as

$$f_X(x;\mu,\sigma,\alpha) = \frac{\alpha}{2\sigma} \frac{[\Gamma(3/\alpha)]^{1/2}}{[\Gamma(1/\alpha)]^{3/2}} \exp\left(-\left|\frac{x-\mu}{A(\alpha,\sigma)}\right|^{\alpha}\right) \qquad x \in \mathbb{R}, \qquad (1)$$

where

$$A(\alpha,\sigma) = \sigma \left[\frac{\Gamma(1/\alpha)}{\Gamma(3/\alpha)}\right]^{1/2} \qquad (2)$$

and $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ $(z > 0)$ is the complete gamma function.

The generalized Gaussian distribution (GGD) is symmetric with respect to μ. $A(\alpha,\sigma)$ is a scaling factor which defines the dispersion of the distribution, hence it is a generalized measure of the variance. $\alpha > 0$ is the shape parameter which describes the exponential rate of decay: heavier tails correspond to smaller values of α.

The generalized Gaussian family includes a variety of random variables. Some well known classes of distributions are generated by a parametrization of the exponential decay of the GGD. When $\alpha = 1$, the GGD corresponds to a Laplacian, or double exponential, distribution. For $\alpha = 2$ one has a Gaussian distribution. When $\alpha \to +\infty$ the GGD converges to a uniform distribution in $(\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma)$, while when $\alpha \to 0^+$ we have an impulse probability function at $x = \mu$.

All the odd central moments of distribution (1) are zero, $\mathbf{E}\,(X-\mu)^r = 0$ ($r = 1,3,5,\ldots$), and the even central moments are

$$\mathbf{E}\,(X-\mu)^r = \left[\frac{\sigma^2\Gamma(1/\alpha)}{\Gamma(3/\alpha)}\right]^{r/2} \frac{\Gamma((r+1)/\alpha)}{\Gamma(1/\alpha)} \qquad r = 2,4,6,\ldots \tag{3}$$

With a straightforward standardization and some reductions from (1), we can obtain the following GGD with zero-mean and unit-variance

$$f_X(x;\alpha) = \frac{\alpha}{2}\frac{[\Gamma(3/\alpha)]^{1/2}}{[\Gamma(1/\alpha)]^{3/2}} \exp\left(-|bx|^{\alpha}\right), \tag{4}$$

with $b = [\Gamma(3/\alpha)/\Gamma(1/\alpha)]^{1/2}$.

In the following, we confine our attention to generalized Gaussian random variables with density (4). For $0 < \alpha < 2$ the density (4) is suitable for modeling many physical and financial processes with heavy tails. It is worth noting that, for GGD with $0 < \alpha < 2$, all the moments are finite (this is not the case for other heavy-tailed densities, like e.g. stable densities).

The kurtosis of distribution (4) is

$$\mathcal{K}(\alpha) = \frac{\Gamma(1/\alpha)\,\Gamma(5/\alpha)}{\left[\Gamma(3/\alpha)\right]^2}, \tag{5}$$

for which the following results hold: $\lim_{\alpha\to 0^+} \mathcal{K}(\alpha) = +\infty$, $\lim_{\alpha\to+\infty} \mathcal{K}(\alpha) = 1.8$ (see Domínguez-Molina & González-Farías, 2002). Figure 1 shows the generalized Gaussian densities for different values of the parameter α, with zero mean and unit variance.

2 Simulating the generalized Gaussian density with $\alpha = 1/2$

With regard to density (4), if we address the case $\alpha = 1/2$ we obtain the GGD

$$f_X(x) = \frac{\sqrt{30}}{2} \exp\left(-2\sqrt{30}\,|x|^{1/2}\right) \tag{6}$$

and the cumulative distribution function

$$F_X(x) = \begin{cases} \frac{1}{2}\left(1+2\sqrt{30}\,|x|^{1/2}\right)\exp\left(-2\sqrt{30}\,|x|^{1/2}\right) & x \le 0 \\ 1 - \frac{1}{2}\left(1+2\sqrt{30}\,|x|^{1/2}\right)\exp\left(-2\sqrt{30}\,|x|^{1/2}\right) & x \ge 0\,. \end{cases} \tag{7}$$

As well known, a large number of realizations of a random variable X with cumulative distribution function (CDF) $F_X(x)$ can be obtained as $x_k = F_X^{-1}(u_k)$, where u_k are random numbers uniform over $[0,1)$. The inverse function of the CDF (7) can be expressed in terms of the *Lambert W function*.

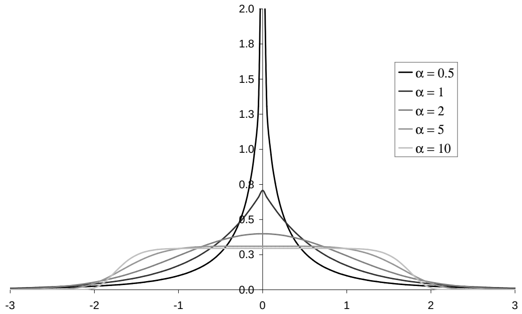

Figure 1. *Generalized Gaussian densities for different values of the parameter* α, *with* $\mu = 0$ *and* $\sigma = 1$.

3 The Lambert W function

The Lambert W function, specified implicitly as the root of the equation

$$W(z)e^{W(z)} = z, \tag{8}$$

is a multivalued function defined in general for z complex and assuming values $W(z)$ complex. If z is real and $z < -1/e$, then $W(z)$ is multivalued complex. If $z \in \mathbb{R}$ and $-1/e \leq z < 0$, there are two possible real values of $W(z)$: the branch satisfying $W(z) \geq -1$ is usually denoted by $W_0(z)$ and called the *principal branch* of the W function, and the other branch satisfying $W(z) \leq -1$ is denoted by $W_{-1}(z)$. If $z \in \mathbb{R}$ and $z \geq 0$, there is a single value for $W(z)$ which also belongs to the principle branch $W_0(z)$. The choice of solution branch usually depends on physical arguments or boundary conditions (for further discussion see Corless *et al.*, 1996).

With some algebra the inverse function of the CDF (7) can be written as

$$F_X^{-1}(u) = \begin{cases} -\frac{1}{2\sqrt{30}}\left[1 + W_{-1}(-2u/e)\right]^2 & 0 < u \leq 1/2 \\ \\ -\frac{1}{2\sqrt{30}}\left[1 + W_{-1}(-2(1-u)/e)\right]^2 & 1/2 \leq u < 1\,. \end{cases} \tag{9}$$

Equation (9) solves the problem of obtaining a large number of realizations of generalized Gaussian density with $\alpha = 1/2$, provided we are able to compute the function W_{-1}.

A high-precision evaluation of the Lambert W function is available in Maple and Mathematica softwares. In particular, Maple computes the real values of W using the third-order Halley's method (see Alefeld, 1981), which in our case ($we^w = x$) gives rise to the following recursion

$$w_{j+1} = w_j - \frac{w_j e^{w_j} - x}{(w_j + 1)\, e^{w_j} - \frac{(w_j+2)(w_j e^{w_j} - x)}{2w_j+2}}\,. \tag{10}$$

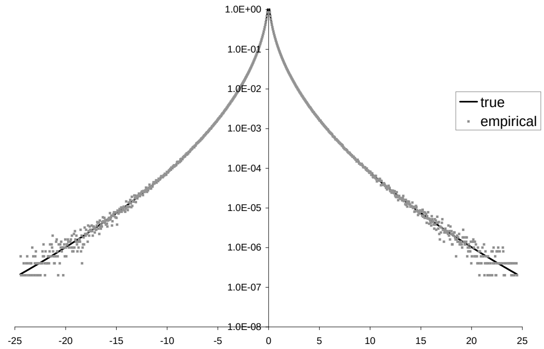

Figure 2. *Estimated probability density function with* $\alpha = 1/2$ *and the theoretical density* (6).

Analytical approximations of the W function are also available and can be used as an initial guess in the iterative scheme (10) (see Corless *et al.*, 1996, and Chapeau-Blondeau & Monir, 2002).

Based on such results, a fast and fairly accurate simulation procedure can be defined. Figure 2 shows the simulated probability density function with $\alpha = 1/2$ and the theoretical density (6).

References

ALEFELD, G. 1981. On the convergence of Halley's method. *The American Mathematical Monthly*, **88**(7), 530–536.

CHAPEAU-BLONDEAU, F., & MONIR, A. 2002. Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. *IEEE Transactions on Signal Processing*, **50**(9), 2160–2165.

CORLESS, R.M., GONNET, G.H., HARE, D.E.G., JEFFREY, D.J., & KNUTH, D.E. 1996. On the Lambert W function. *Advances in Computational Mathematics*, **5**, 329–359.

DOMÍNGUEZ-MOLINA, J.A., & GONZÁLEZ-FARÍAS, G. 2002. Sobre la estimación del parámetro de forma de la distribución gausiana generalizada. *Aportaciones Matemáticas.* Serie Comunicaciones, **30**, 117–120.

KOKKINAKIS, K., & NANDI, A.K. 2005. Exponent parameter estimation for a generalized Gaussian probability density functions with application to speech modeling. *Signal Processing*, **85**, 1852–1858.