APPROXIMATING BY ITERATED FUNCTION SYSTEMS AND ITERATED MULTIFUNCTION SYSTEMS

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ABSTRACT: We recall the theory of iterated function systems and iterated multifunction systems. Some applications in fractal estimation and fractal simulation are given.

KEYWORDS: Iterated function systems, fractals, estimation, Brownian motion

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In this paper we recall some basic facts on *iterated function systems* (IFS) and on *iterated multifunction systems* (IMS) over a metric space and we show some applications of these in fractal estimation and simulation. Hutchinson (1981) and Barnsley and Demko (1985) showed how systems of contractive maps with associated probabilities – called "iterated function systems" by the latter – acting in a parallel manner either deterministically or probabilistically, can be used to construct fractal sets and measures. Many applications in economics and finance involves these types of dynamical systems; in Iacus and La Torre (2005a, 2005b, 2006) one can find fractal approximations of distribution and density functions, fractal simulations of Brownian motions and stochastic processes. IMS are a generalization of IFS, from a standard point-to-point contraction mapping to a set-valued operator. IMS operators T are defined by the parallel action of a set of contractive multifunctions T_i. Under suitable conditions T is contractive, implying the existence of a fixed-point multifunction $\bar{x}$ such that $\bar{x} \in T\bar{x}$. The multifunction T satisfies the following contractivity condition: there exists a $c \in [0,1)$ such that $d_h(Tx,Ty) \leq cd(x,y)$ for all $x,y \in X$, where d_h denotes the Hausdorff metric. From a fundamental theorem of Covier and Naylor, if T is contractive in the above sense, then there exists a fixed point $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$. Note that $\bar{x}$ is not necessarily unique. The set of fixed points of T, to be denoted as X_T, plays an important role in application. A corollary of the Covier-Naylor theorem, based on projections onto sets, is a method to construct solutions to the fixed point equation $x \in Tx$, essentially by means of an iterative method that converges to a point $x \in X_T$. For IMS it is possible to show two results that can be viewed as multifunction analogues of those that apply when T is a contractive point-to-point mapping (in which case Banach's fixed point theorem applies), namely: (i) a continuity property of fixed point sets X_T and (ii) "collage theorems" for multifunctions. These results are important in the inverse problem of approximation and for application by fixed points of contractive mappings, which we state for the case in which $T : X \to X$ is a point-to-point contraction mapping: given a "target" element $y \in X$, we seek a contraction mapping T with fixed point $\overline{x}$ such that $d(y,\overline{x})$ is as small as possible. In practical applications, however, it is difficult to construct solutions to this problem. Instead, one relies on the following simple consequence of Banach's fixed point theorem,

$$d(y,\overline{x}) \leq \frac{1}{1-c} d(y,Ty), \tag{1}$$

where c is the contractivity factor of T. In fractal based applications this result is known as the "collage theorem". Instead of trying to minimize the approximation error $d(y,\overline{x})$, one searches for a contraction mapping T that minimizes the *collage error* $d(y,Ty)$. Analogous results can be shown for IMS.

1 Iterated function systems and iterated multifunction systems

In the following we let $d(x,y)$ denote the Euclidean distance. We shall also let $\mathcal{H}(X)$ denote the space of all compact subsets of X and $d_h(A,B)$ the Hausdorff distance between A and B, that is

$$d_h(A,B) = \max\{\max_{x\in A} d'(x,B), \max_{x\in B} d'(x,A)\}, \tag{2}$$

where $d'(x,A)$ is the usual distance between the point x and the set A, i.e. $d'(x,A) = \min_{y\in A} d(x,y)$. In the following we will denote by $h(A,B) = \max_{x\in A} d'(x,B)$. It is well known that the space $(\mathcal{H}(X),d_h)$ is a complete metric space if X is complete. First of all we introduce the idea of an iterated function system. Once again, (X,d) denotes a complete metric space, typically $[0,1]^n$. Let $\{w_1,\cdots,w_N\}$ be a set of contraction maps $w_i : X \to X$, to be referred to as an N-map IFS. Let $c_i \in [0,1)$ denote the contraction factors of the w_i and define $c = \max_{1\leq i\leq N} c_i \in [0,1)$. As before, we let $\mathcal{H}(X)$ denote the set of nonempty compact subsets of X and h the Hausdorff metric. Associated with the IFS maps w_i is a set-valued mapping $w : \mathcal{H}(X) \to \mathcal{H}(X)$ the action of which is defined to be $w(S) = \bigcup_{i=1}^N w_i(S)$, $S \in \mathcal{H}(X)$, where $w_i(S) := \{w_i(x), x \in S\}$ is the image of S under w_i, $i = 1,2,\cdots,N$. It is a standard result that w is a contraction mapping on $(\mathcal{H}(X),d_h)$, that is $d_h(w(A),w(B)) \leq c d_h(A,B)$, $A,B \in \mathcal{H}(X)$. Consequently, there exists a unique set $A \in \mathcal{H}(X)$, such that $w(A) = A$, the so-called *attractor* of the IFS. The equation $A = w(A)$ obviously implies that A is self-tiling, i.e. A is union of (distorted) copies of itself. Moreover, for any $S_0 \in \mathcal{H}(X)$, the sequence of sets $S_n \in \mathcal{H}(X)$ defined by $S_{n+1} = w(S_n)$ converges in Hausdorff metric to A. As extension of IFS, consider a set of $T_i : X \rightrightarrows X$ of multifunctions where $i \in 1\ldots n$ and $T_i x \in \mathcal{H}(X)$ for all i. We now construct the multifunction $T : X \rightrightarrows X$ where $Tx = \bigcup_i T_i x$. Suppose that the multifunctions T_i are contractions with contractivity factor $c_i \in [0,1)$, that is, $d_h(T_i x, T_i y) \leq c_i d(x,y)$. From the Covier-Nadler theorem cited earlier, there exists a point $\overline{x} \in T\overline{x}$. Now given a compact set $A \in \mathcal{H}$ consider the image $T(A) = \bigcup_{a\in A} Ta \in \mathcal{H}(X)$. Since $T : (X,d) \to (\mathcal{H}(X), d_h)$ is a continuous function then $T(A)$ is a compact subset of $\mathcal{H}(X)$. So we can build a multifunction $T^* : \mathcal{H}(X) \rightrightarrows \mathcal{H}(X)$ defined by $T^*(A) = T(A)$ and consider the Hausdorff distance on $\mathcal{H}(X)$, that is given two subset $A,B \subset \mathcal{H}(X)$ we can calculate

$$d_{hh}(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d_h(x,y), \sup_{x\in A}\inf_{y\in B} d_h(x,y)\}. \tag{3}$$

We have that $T^* : \mathcal{H}(X) \rightrightarrows \mathcal{H}(X)$ and $d_{hh}(T^*(A), T^*(B)) \le c d_h(A,B)$. Now given a point $x \in X$ and a compact set $A \subset X$ we know that the function $d(x,a)$ has at least one minimum point $\bar{a}$ when $a \in A$. We call $\bar{a}$ the *projection* of the point x on the set A and denote it as $\bar{a} = \pi_x A$. Obviously $\bar{a}$ is not unique but we choose one of the minima. We now define the following projection function P associated with a multifunction T defined as $P(x) = \pi_x(Tx)$. We therefore have the following result (see Kunze, La Torre and Vrscay 2006).

Theorem 1.1. *Let (X,d) be a complete metric space and $T_i : X \to \mathcal{H}(X)$ be a finite number of contractions with contractivity factors $c_i \in [0,1)$. Let $c = \max_i c_i$. Then*

1. *For all compact $A \subset X$ there exists a compact subset $\bar{A} \subset X$ such that $A_{n+1} = P(A_n) \to \bar{A}$ when $n \to +\infty$.*
2. *$\bar{A} \subset \bigcup_i T_i(\bar{A})$.*

2 Application of IFS and IMS

Figures 1 and 2 show some examples in which IFS and IMS are used for approximation purposes. In particular figure 1 shows how they can be used for estimation of small and medium samples; numerical examples show that this estimator works better than the empirical distribution function (EDF). More details can be found in Iacus and La Torre (2005a, 2005b). Figure 2 shows how to approximate a classical Brownian motion; also in this case the attractor works better than Eulero scheme for simulation (see Iacus and La Torre (2006) and the references therein).

References

BARNSLEY, M., ERVIN V. HARDIN D., & LANCASTER, J. 1985. Solution of an inverse problem for fractals and other sets. *Proc. Natl. Acad. Sci.*, **83**, 1975–1977.

FORTE, B., MENDIVIL F., & VRSCAY, E.R. 1999. IFS operators on integral transforms. *Fractals: Theory and Applications in Engineering*.

HUTCHINSON, J. 1981. Fractals and self-similarity. *Indiana Univ. J. Math.*, **30**, 713–747.

IACUS, S.M., & LA TORRE, D. 2005a. Approximating distribution functions by iterated function systems. *Journal of Applied Mathematics and Decision Sciences*, **1**, 334–345.

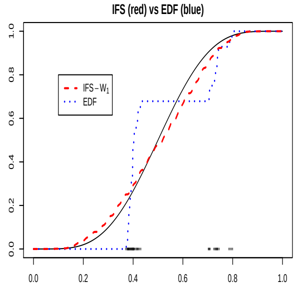

Figure 1. *IFS approximations of distribution functions*

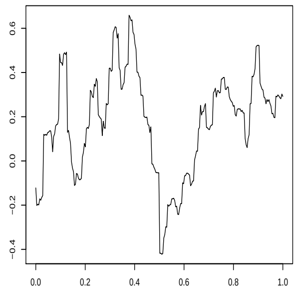

Figure 2. *IFS simulation of Brownian motion*

IACUS, S.M., & LA TORRE, D. 2005b. A comparative simulation study on the IFS distribution function estimator. *Nonlinear Analysis: Real World Applications*, **6**, 774–785.

IACUS, S.M., & LA TORRE, D. 2006. IFSM representation of Brownian motion with applications to simulation. *Berkeley UNIMI Research Papers*.

KUNZE, H., LA TORRE D., & VRSCAY, ED. 2006. Contractive multifunctions, fixed point inclusions and iterated multifunction systems. *Berkeley UNIMI Research Papers*.

LA TORRE, D., MENDIVIL F., & VRSCAY, ED. 2006. IFS on multifunctions. *Berkeley UNIMI Research Papers (forthcoming in Proceedings of Math Everywhere, Springer)*.