A fractional differo-integral approach for fractal compound financial laws

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Abstract

In this work we determine the financial laws following which the risk-less component of a risky portfolio must evolve in order of avoid the possibility of arbitrages when the dynamics of the stochastic component of the portfolio is driven by a fractional Brownian motion. In order to deal with this problem we specify a deterministic fractional differential equation and we solve it using Liouville's second method.

Keywords

Financial laws, arbitrage, fractional Brownian motion, fractional differo-integral calculus, Liouville's second method.

1. Introduction

Since the pioneering works of Bachelier and Working, the classical approach concerning the analysis of financial asset returns has been characterized by assumptions of independent and identical probability distribution for the random variables

$$\ln[P(t+\Delta t)]-\ln[P(t)], \text{ with } t=0,\ 1,\ \ldots,\ N-\Delta t,$$

where $P(t)$ is the financial asset price at time t.

In particular, in the modern economic-financial theories (such as, for instance, those on the impossibility of making arbitrage and on derivative pricing) the specific hypothesis of independent probability distribution has became a paradigm.

Nevertheless, an increasing number of empirical analyses have highlighted the partial inadequacy of the assumption of the latter hypothesis, because of the continuous presences of self-dependence both in the short-middle term and in the long one.

Starting from these peculiarities, several Authors have rejected the hypothesis of efficiency in weak form for the financial markets and They have conjectured the non markovianity for the stochastic processes generating variations of financial asset prices.

Such empirical results have favoured an increasing interest towards the stochastic processes known as "fractional Brownian motions" because of their capability of representing Gaussian stochastic processes whose generic increments are stochastically dependent. In approximate terms, such stochastic processes can be interpreted as a properly weighted summation of the past realizations of a standard Brownian motion. It is possible to formalize this characterization of the fractional Brownian motion by means of the stochastic differo-integral calculus:

$$B_H(t)=\frac{1}{\Gamma(H+0.5)}\left[\int_{-\infty}^{0}\left\{(t-s)^{H-0.5}-(-s)^{H-0.5}\right\}dB(s)+\int_{0}^{t}(t-s)^{H-0.5}dB(s)\right]$$

where $B_H(t)$ is the fractional Brownian motion at time t, with

$$H\begin{cases}\in(0,0.5)\text{ if } B_H(t)\text{ is characterized by negative long-term dependence}\\ =0.5\text{ if } B_H(t)\text{ is characterized by independence}\\ \in(0.5,1)\text{ if } B_H(t)\text{ is characterized by positive long-term dependence}\end{cases},$$

$\Gamma(\cdot)$ is the gamma function, and $B(t)$ is the standard Brownian motion at time *t*.

In such a way, it is possible to see the fractional Brownian motion as a differo-integral of the standard Brownian motion with parameter $H-0.5$.

It is to notice that, above all in the last years, there has been a growth in interest in the possible application of the fractional Brownian motion in finance. In particular, the introduction of this stochastic process in the financial framework has strong implications because it permits the presence of arbitrage possibilities and, in its turn, the arbitrage implies the impossibility of pricing important financial tools like the options. In the majority of the works known to us such a problem is faced by modifying the structure of this process in order to obtain a standard-like model for option pricing, or a standard-like formula of differential stochastic calculus, so avoiding possibilities of arbitrage.

On the contrary, in this work we assume that the dynamics of the stochastic component of a risky portfolio is driven by a fractional Brownian motion (that we leave unchanged) and we determine a new financial law following which the risk-less component of this portfolio must evolve in order to avoid any possibility of arbitrages. In particular, the financial law we propose generalizes the classical one, and ensures the possibility of pricing financial tools.

In more detail, in order to describe such a dynamics of the risk-less component of the risky portfolio,

- firstly, we start from the following fractional finite difference equation which generalizes the classical hypothesis for the compound interest rate financial law:
$$\Delta^{\alpha}C(t)\overset{\text{def}}{=}C\left(t+\Delta t^{\alpha}\right)-C(t)=\delta\cdot C(t)\cdot\Delta t^{\alpha}+o\left(\Delta t^{\alpha}\right)$$
where $\alpha\in(0,\ 1)\cap\mathbf{Q}$ is the fractional order of differentiation, $C(t)$ is the risk-less component of the immunized portfolio at time *t*, $\delta=\ln(1+i)$ is the instantaneous interest intensity, *i* is the periodical interest rate, and $o(\cdot)$ has the usual meaning of infinitesimal function;
- then, by means of an application of the Letnikov's results (see, for details, [Samko *et al.*, 1993]), we obtain the following fractional differential equation:
$$\lim_{\Delta t\to 0^{+}}\frac{C\left(t+\Delta t^{\alpha}\right)-C(t)}{\Delta t^{\alpha}}=\frac{d^{\alpha}}{dt^{\alpha}}C(t)=\delta\cdot C(t)+\lim_{\Delta t\to 0^{+}}\frac{o\left(\Delta t^{\alpha}\right)}{\Delta t^{\alpha}},$$
$$\text{i.e. }\frac{d^{\alpha}}{dt^{\alpha}}C(t)=\delta\cdot C(t);\tag{1}$$
- again, by means of an original application of the differo-integral calculus (outlined in the next section), we give the solution of fractional differential equation (1);
- finally, in section **3.** we present some remarks concerning the role played by the arbitrary constants of this solution in order to avoid any possibility of arbitrages.

2. The solving approach: an outline

Fractal differo-integral calculus is a particular sort of calculus in which the order of differo-integration is a real number. In this paper, $\frac{d^{\alpha}}{dx^{\alpha}}[\cdot]$, with $\alpha\in\mathbf{R}$, represents the differo-integral Riemann-Liouville operator, in particular, if $\alpha\begin{cases}<0\text{ one has integration}\\>0\text{ one has differentation}\end{cases}$.

Although several definitions of fractional integration and derivation exist, all of them have to obey the following usual basic rules:

- if $\alpha = 0$ then $\frac{d^{\alpha}}{dx^{\alpha}} f(x) = f(x)$,
- $\frac{d^{\alpha}}{dx^{\alpha}}\left[a \cdot f(x) + b \cdot g(x)\right] = a \cdot \frac{d^{\alpha}}{dx^{\alpha}} f(x) + b \cdot \frac{d^{\alpha}}{dx^{\alpha}} g(x)$, with $a, b \in \mathbf{R}$;
- $\frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{\beta}}{dx^{\beta}} f(x)\right] = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}} f(x) + g(x)$, with $\beta \in \mathbf{R}$, where $g(x)$ is a suitable function (see, for instance, [Miller *et al.*, 1993] and [Podlubny, 1999]).

With regard to the solving procedure, we use a method - we have autonomously developed - which is inspired to the one proposes [Miller *et al.*, 1993]. It can be summarized as follows:

step 0: one sets $\alpha \in (0,\ 1) \cap \mathbf{Q}$, that is $0 < \alpha = m/n < 1$, with $m,\ n \in \mathbf{N}^{+}$ such that m and n without common divisors; then in (1)

$$\frac{d^{\alpha}}{dt^{\alpha}} C(t) = \frac{d^{m/n}}{dt^{m/n}} C(t) = \delta \cdot C(t);$$

step 1: one applies the operator $\frac{d^{m/n}}{dt^{m/n}}[\cdot]$ to both the terms of the differential equation, that is

$$\frac{d^{m/n}}{dt^{m/n}}\left[\frac{d^{m/n}}{dt^{m/n}} C(t)\right] = \frac{d^{m/n}}{dt^{m/n}}\left[\delta \cdot C(t)\right],$$

from which, by the property of the operator $\frac{d^{m/n}}{dt^{m/n}}[\cdot]$ (first term), and by the application of the Liouville's second method (second term), one obtains:

$$\frac{d^{2 \cdot m/n}}{dt^{2 \cdot m/n}} C(t) = \delta \cdot \frac{d^{m/n}}{dt^{m/n}} C(t) + c_1 \cdot t^{-m/n-1};$$

step 2: recalling from **step 0** that $\frac{d^{m/n}}{dt^{m/n}} C(t) = \delta \cdot C(t)$, and substituting it in **step 1**, one obtains

$$\frac{d^{2 \cdot m/n}}{dt^{2 \cdot m/n}} C(t) = \delta \cdot \left[\delta \cdot C(t)\right] + c_1 \cdot t^{-m/n-1};$$

step 3: iterating other $n-2$ times **step 1**-**step 2**, one has

$$\underbrace{\frac{d^{m/n}}{dt^{m/n}}\left\{\frac{d^{m/n}}{dt^{m/n}} \cdots\right.}_{n-2 \text{ times}}\left.\left[\frac{d^{2 \cdot m/n}}{dt^{2 \cdot m/n}} C(t)\right] \cdots\right\} = \delta^{n} \cdot C(t) + \sum_{j=1}^{n-1} c_j \cdot \frac{d^{(n-1-j)\cdot(m/n)}}{dt^{(n-1-j)\cdot(m/n)}} t^{-m/n-1};$$

step 4: it is possible to prove that:

$$\frac{d^{(n-1-j)\cdot(m/n)}}{dt^{(n-1-j)\cdot(m/n)}} t^{-(m/n)-1} = \frac{(-1)^{(n-1-j)\cdot(m/n)} \cdot \Gamma\left((m/n)\cdot(n-j)+1\right)}{\Gamma(m/n+1)} t^{-(m/n)\cdot(n-j)-1};$$

moreover, we define

$$k_j \overset{\text{def}}{=} \frac{(-1)^{(n-1-j)\cdot(m/n)} \cdot \Gamma\left((m/n)\cdot(n-j)+1\right)}{\Gamma(m/n+1)};$$

step 5: finally, substituting **step 4** in **step 3** one obtains the following integer-order linear differential equation:

$$\frac{d^{m}}{dt^{m}} C(t) - \delta^{n} \cdot C(t) = \sum_{j=1}^{n-1} c_j \cdot k_j \cdot t^{-(m/n)\cdot(n-j)-1};$$

step 6: the solution is the fractional compound interest rate financial law:

$$C(t)=\underbrace{\sum_{j=1}^{m}\tilde{c}_j\cdot\exp(\lambda_j\cdot t)}_{\text{solut. of the homog. diff. eq.}}+\underbrace{\sum_{j=1}^{n-1}c_j\cdot k_j\cdot\exp(t)\cdot\int_0^t\exp(-y)\cdot y^{-(m/n)\cdot(n-j)-1}dy}_{\text{particular solution of the non homogeneous differential equation}},$$

where $\tilde{c}_j$ and c_j are $m+n-1$ arbitrary constants, and λ_j are the solutions of the equation $\lambda^n-\delta^n=0$.

It is to notice that the use of the presented solving approach ensures the existence of $m+n-1$ arbitrary constants (and we need several arbitrary constants to suitably adapt the financial law to the investigated financial environment), whereas, as $0<\alpha<1$, a Mittag-Leffler function-based solving approach should give only 1 arbitrary constant.

3. The case *m*=1 and *n*=2

Let $\frac{d^{1/2}}{dt^{1/2}}C(t)=\delta\cdot C(t)$ be the starting fractional-order differential equation. Applying the solving procedure presented in section **2.**, one obtains the following solution of the later differential equation, that is the fractional capitalization law:

$$C(t)=\underbrace{\tilde{c}\cdot\exp(\delta^2\cdot t)}_{\text{sol. of the homog. diff. eq.}}+\underbrace{c_1\cdot k_1\cdot\exp(t)\cdot\int_0^t\exp(-y)\cdot y^{-3/2}dy}_{\text{particular solut. of the non homogeneous diff. equation}}$$

where $\tilde{c}$ and c_1 are arbitrary constants.

Concerning the particular solution of the non homogeneous differential equation, one obtains:

$$\int_0^t\exp(-y)\cdot y^{-3/2}dy=-\frac{2}{\sqrt{y}}\exp(-y)-2\cdot\sqrt{\pi}\cdot\operatorname{erf}\left(\sqrt{y}\right)\Bigg|_0^t$$

where $\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z\exp(-u^2)du$ is the so-called "error function".

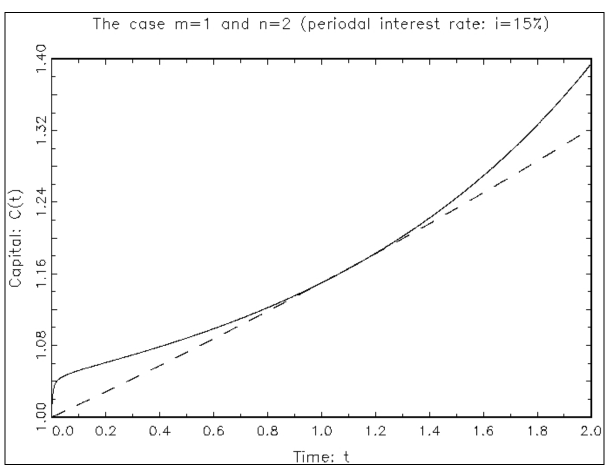

Figure 1

Finally, with regard to the arbitrary constants:

$$\begin{cases}\text{no-arbitrage possibility at time } t=0:\ C(0)=1\\ \text{no-arbitrage possibility at time } t=1:\ C(1)=\exp(\delta)\end{cases},$$

from which

$$\begin{cases} \tilde{c} = 1 \\ c_1 = \dfrac{\exp\left(\delta^2\right) - \exp(\delta)}{2 \cdot \exp(1) \cdot k_1 \cdot a}, \text{ with } a = \dfrac{\exp(-y)}{\sqrt{y}} + \sqrt{\pi} \cdot \operatorname{erf}(y)\Big|_0^1 \end{cases}.$$

In figure 1 we give a graphical representation of the considered "case m=1 and n=2" (continuous line) *vs.* the classical case (dotted line) by using a periodical interest rate i=15%.

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