A liability adequacy test for mathematical provision

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In a fair valuation context, let us introduce two probability spaces $(\Omega, \mathcal{F}', P')$, $(\Omega, \mathcal{F}'', P'')$, where $\mathcal{F}'$ and $\mathcal{F}''$ are the σ-algebras containing, respectively, the *financial events* and the *life duration events*. Given the independence of the mortality fluctuations on the interest rates randomness, we denote by $(\Omega, \mathcal{F}, P)$ the probability space canonically generated by the preceding two. $\mathcal{F}$ contains the information flow about both mortality and financial history, represented by the filtration $\{\mathcal{F}_k\} \subseteq \mathcal{F}(\mathcal{F}_k = \mathcal{F}'_k \cup \mathcal{F}''_k$ with $\{\mathcal{F}'_k\} \subseteq \mathcal{F}'$ and $\{\mathcal{F}''_k\} \subseteq \mathcal{F}'')$.

Assuming a frictionless market, with continuous trading, no restrictions on borrowing or short-sales, the zero-bond and the stocks being both infinitely divisible, the fair value of the reserve at time t of a portfolio of life insurance contracts - with obvious meaning of the symbol $\mathcal{F}_t$- is given by

$$V_t = E\left[\sum_{r>t} CF_r v(t,r)/F_t\right], \tag{1}$$

where $v(t,r)$ is the present value at time t of one monetary unit due at time r, CF_r is the net cash flow at time r, and E represents the expectation under the risk-neutral probability measure, whose existence derives by well known results, based on the completeness of the market.

With respect to the demographic risk context, given that the demographic valuation is not supported by the hypotheses of completeness of the market, the current valuation can be represented by means of the expectation consistently with the best prediction of the demographic scenario. In a general perspective, a fair valuation procedure involves the latest information on the two main factors bringing risk to the business, properly interest rates and mortality (see [CDLOS]).

Formula (**??**) can be easily specialised in the case of a portfolio of different life annuities with benefits payable at the beginning of each period. In this case we split the portfolio in m homogeneous sub-portfolios characterized by common aspects, that is age at issue, policy duration, payments, deferment period, etc.

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Let us introduce the following notations:

$$n = \text{maximum term for all contracts},$$

$$S_{i,r} = \text{number of survivors at time } r \text{ of the insureds of the } i\text{-th group},$$

$$L_i = \text{constant annual benefit for each contract of the } i\text{-th group},$$

$$P_{i,r} = \text{premium paid at time } r \text{ for each contract of the } i\text{-th group},$$

$$T_i = \text{deferment period } (T_i = 0, 1, \ldots),$$

$$\tau_i = \text{premium payment period } (0 \leq \tau_i < T_i).$$

Hence formula (**??**) becomes

$$V_t = E\left[\sum_{r=t+1}^{m}\sum_{i=1}^{m}\left[S_{i,r}\left(L_i \mathbf{1}_{(n_i \geq r)\wedge(r \geq T_i)} - P_{i,r}\mathbf{1}_{(n_i \geq r)\wedge(r<\tau_i)}\right)\right] v(t,r)/F_t\right], \tag{2}$$

where the indicator function $\mathbf{1}_{(n_i \geq r)\wedge(r \geq T_i)}$ takes the value 1 if $n_i \geq r$ and $r \geq T_i$, 0 otherwise, whilst the indicator function $\mathbf{1}_{(n_i \geq r)\wedge(r<\tau_i)}$ takes the value 1 if $n_i \geq r$ and $r < \tau_i$, 0 otherwise.

Formula (**??**) is framed in a *forward* perspective, within a current valuation provided at the initial position in 0. Analogously this formula can be re-interpreted in a *spot* perspective, according to a year by year valuation, that is providing the current value of the reserve at the end of the year, valued at the beginning of the year itself (cf. [CDLOS]).

In a solvency assessment perspective, models involving the so-called *quantile reserve* play a fundamental role because of their specific links to the *Value-at-Risk*.

Indicating by $R(t)$ the financial position at time t, that is, in this case, the stochastic mathematical provision of a portfolio of contracts, the quantile reserve at confidence level α $(0 < \alpha < 1)$, is the value $R^*_\alpha(t)$ implicitly defined by the following equation:

$$P\{R(t) > R^*_\alpha(t)\} = 1 - \alpha. \tag{3}$$

Moreover, considering the time interval $[t, t+h]$ and the financial positions at its extremes, say $r(t)$ and $R(t+h)$ respectively, the potential periodic loss is defined as (Teugels et al. 2002):

$$L = r(t) - R(t+h), \tag{4}$$

therefore at confidence level α, the Value-at-Risk $VaR(\alpha)$ is given by:

$$P\{L > VaR(\alpha)\} = 1 - \alpha \Leftrightarrow VaR(\alpha) = F^{-1}(\alpha), \tag{5}$$

F being the cumulative distribution function of $R(t)$.

In this section we propose a simulation procedure to quantify the VaR of two homogeneous portfolios of deferred life annuities. For sake of simplicity, the valuation of the financial instruments composing the ZCB portfolio will be made assuming a term structure of interest rates based on a square root CIR process:

$$dr_i = -\alpha(r_i - \mu)dt + \sigma\sqrt{r_i}dW_t,$$

with α and σ positive constants, μ the long term mean and W_t a Wiener process.

Referring to the mortality scenario, in a fair valuation estimating framework, we consider a marking-to-model system for the demographic quantities. We assume that the best prediction for the time evolution of the surviving phenomenon is represented by a fixed set of survival probabilities, estimated taking into account the improving trend of mortality rates (best estimate).

For each portfolio, we simulate 100000 values of the potential periodic loss L defined in §2. The simulated $\{L(j)\})$, $j = 1, 2, \ldots, 100000$ can be treated as a sample from a normal distribution (cf. [CDLOS]), which we use to estimate the VaR. The results obtained for a portfolio of 1000 deferred 15-years temporary unitary life annuities (deferment period $T = 5$), for policyholders aged 30 at issue. Periodic premiums are paid at the beginning of each year of the deferment period.

We compute the VaR , during the deferment period. We frame the calculation within a spot perspective, according to a year by year valuation.

Under the above hypothesis about survival and rates, we present the results obtained for a portfolio of 1000 deferred 10-years temporary unitary life annuities.

Tables supplying the maximum future net carrying amount of the final reserve within a set confidence interval will be provided.

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