ON CHARACTERIZATION OF CONVEX PREMIUM PRINCIPLES

Marta Cardin[1] and Graziella Pacelli[2].

[1] Dipartimento di Matematica Applicata
Università Cà Foscari di Venezia
(e-mail: mcardin@unive.it)

[2] Dipartimento di Scienze Sociali
Università Politecnica delle Marche
(e-mail: g.pacelli@univpm.it)

ABSTRACT: In actuarial literature the properties of risk measures or insurance premium principles have been extensively studied . We propose a characterization of a particular class of coherent risk measures . The considered premium principles are obtained by expansion of TVar measures.

KEYWORDS: Risk measures, premium principles, Choquet measures distortion function,TVar

Premium principles are the most important risk measures in actuarial sciences. There are different methods that actuaries use to develop premium principles. In this paper we follow a characterization method to represent a class of premium principles connected with Wang's premium principles. Wang's approach is related to the coherent risk measures (Artzner *et al.* , 1999) is founded in Yaari's dual theory of risk and is connected with non -additive measure theory (Denneberg, 1994).We give a representation for some premium principles by a convex combination of particular risk measures.

First of all we discuss desirable properties of premium principles. In actuarial applications a risk is represented by a nonnegative random variable. We consider an insurance contract in a specified time period $[0,T]$. Let Ω be the state space and $\mathcal{F}$ the event σ-field at the time T. Let $\mathbf{P}$ be a probability measure on $\mathcal{F}$. We consider an insurance contract described by a non-negative random bounded variable X, $X : \Omega \to \mathbb{R}$ where $X(\omega)$ represents its payoff at time T if state ω occurs.

We denote by F_X the distribution function of X i.e. $F_X(x) = \mathbf{P}(\omega \in \Omega : X(\omega) \leq x)$, $x \in \mathbb{R}$, and by S_X the survival function.

Frequently an insurance contract provides a franchise and then it is interesting to consider the values ω such that $X(\omega) > a$: in this case the contract pays for $X(\omega) > a$ and nothing otherwise. Then it is useful to consider also the random variable

$$(X-a)_+ = \max(X(\omega)-a,0)$$

We consider a set, L of nonnegative random variables such that L has the following property:

i) $aX, \quad (X-a)_+, \quad (X-(X-a)_+) \in L \qquad \forall X \in L$, and $a \in [0,+\infty)$.

We denote the insurance prices of the contracts of L by a functional H where

$$H : L \to \widetilde{\mathbb{R}}$$

and $\widetilde{\mathbb{R}}$ is the extended real line. Usually some properties that it is reasonable assume for an insurance functional price H are:

$(\mathbf{P1})$ $H(X) \geq 0$ for all $X \in L$.

Property $(\mathbf{P1})$ is a very natural requirements.

$(\mathbf{P2})$ If $c \in [0, +\infty)$ then $H(c) = c$.

Property $(\mathbf{P2})$ implies that when there is no uncertainty, there is no safety loading.

$(\mathbf{P3})$ $H(X) \leq \sup_{\omega \in \Omega} X(\omega)$ for all $X \in L$.

This is just a natural price condition for any customer who wants to underwrite an insurance policy.

$(\mathbf{P4})$ $H(aX + b) = aH(X) + b$ for all $X \in L$ such that $aX + b \in L$ with $a, b \in [0, +\infty)$.

This is a linearity property.

$(\mathbf{P5})$ If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ for $X, Y \in L$ then $H(X) \leq H(Y)$.

This condition states that the price of the larger risk must be higher.

$(\mathbf{P6})$ $H(X + Y) \leq H(X) + H(Y)$ for all $X, Y \in L$ such that $X + Y \in L$.

This property require the premium for the sum of two risks is not greater that the sum of the individual premiuum; otherwise the buyer would simply insure the two risks separately.

We observe that the properties $(\mathbf{P4})$ and $(\mathbf{P6})$ imply the following property:

$(\mathbf{P7})$ $H(aX + (1-a)Y) \leq aH(X) + (1-a)H(Y)$ for all $X, Y \in L$ and $a \in [0, 1]$ such that $aX + (1-a)Y \in L$.

Convexity means that diversification does not increase the total risk. In the insurance context this property allows for pooling of risks effects.

We present some assumptions which frequently are in the applications :
$(\mathbf{A1})$ $H(X) = H(X - (X-a)_+) + H((X-a)_+)$ for all $X \in L$ and $a \in [0, +\infty)$.

This condition splits into two comonotonic parts a risk X and permits to identify the part of premium charged for the risk with the reinsurance premium charged by the reinsurer.

$(\mathbf{A2})$ If $E(X-a)_+ \leq E(Y-a)_+$ for all $a \in [0,+\infty)$ then $H(X) \leq H(Y)$ for all $X,Y \in L$.
In other words the functional price H respects the stop-loss order. We remember that stop-loss order considers the weight in the tail of distributions; when other characteristics are equals, stop-loss order select the risk with less heavy tails.

$(\mathbf{A3})$ The price, $H(X)$, of the insurance contract X depends only on its distribution F_X.

The hypotheses $(\mathbf{A3})$ says that it is not the state of the world to determine the price of a risk, but the probability distribution of X assigns the price to X. So risks with identical distributions have the same price.
Finally, we present a continuity property that is usual in characterizing certain premium principles.
$(\mathbf{A4})$ $\lim_{n\to+\infty} H(X-(X-n)_+) = H(X)$ for all $X,Y \in L$.

The representation for a class of premium principles is obtained by an integral representation with non-additive measures (Denneberg, 1994) and by distorted probability measures as introduced in (Wang, 1996). In particular we give a modified version of Greco Theorem for the functional H which satisfies some assumptions of the list above on the space of non-negative random variables.

Theorem 1 (Modified Greco Theorem) Let L be a set of nonnegative random variables such that L has property i) and we suppose that $I_\Omega \in L$ where I_Ω is the indicator function of Ω. We consider a premium principle $H : L \to \widetilde{\mathbb{R}}$ such that:

a) $H(I_\Omega) < +\infty$,
b) H satisfies the hypotheses $(\mathbf{A1})$,$(\mathbf{A2})$, $(\mathbf{A4})$.

Then there exists a capacity $\upsilon : 2^\Omega \to \mathbb{R}$ such that for all $X \in L$

$$H(X) = \int_\Omega X d\upsilon = \int_0^{+\infty} \upsilon\{\omega : X(\omega) > x\} dx$$

Corollary 1 Let L be a set of nonnegative random variables such that L has property i). Let H a premium principle that satisfies the hypotheses of Theorem 1, and verifies the property $(\mathbf{P6})$, then the capacity is convex.

It is known, that given a non negative random variable X, for any increasing function g, with $g(0)=0$ and $g(1)=1$, we can define a premium principle

$$H(X)=\int_0^{+\infty} g(S_X(t))dt=\int_\Omega X d\upsilon$$

where g is a distortion function, $\upsilon = go\mathbf{P}$.
Finally we obtain the integral representation result for the considered class of premium principles and the characterization as convex combination of $TVaR_\alpha$.

Theorem 2
Let L be a set of nonnegative random variables such that L has property i) and we suppose that $I_\Omega \in L$ where I_Ω is the indicator function of Ω. We consider a premium principle $H : L \to \widetilde{\mathbb{R}}$ such that:

a) $H(I_\Omega) < +\infty$,
b) H satisfies the hypotheses (**A1**),(**A2**), (**A3**), (**A4**).

Then there exists a probability measure m on $[0,1]$ such that:

$$H(X)=\int_0^1 TVaR_\alpha(X)dm(\alpha)$$

The results obtained for the class of insurance functional prices seems interesting both because the class of functionals is determined from few natural properties and these functional prices follow closely linked together to a well known risk measure as $TVaR_\alpha$, $\alpha \in [0,1]$. Moreover we point out that the most important properties for a functional price follow easily from the obtained representation.

References

ARTZNER, P., DELBAEN, F., EBER, J., & HEATH, D. 1999. Coherent measures of risks. *Mathematical Finance.*, **9**, 203–228.

DENNEBERG, D. 1994. *Non-additive measure and integral*. Dordrecht/Boston/London: Kluwer Academic Publishers.

GRECO, S. 1982. Sulla rappresentazione di funzionali mediante integrali. *Rendiconti Seminario Matematico Università Padova.*, **66**, 21–42.

WANG, S. 1996. Premium Calculation by Trasforming the Layer Premium Density. *Astin Bulletin.*, **26**, 71–92.

WANG, S., YOUNG, V., & PANJER, H. 1997. Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics.*, **21**, 173–183.