ON BOUNDS FOR CONCAVE DISTORTION RISK MEASURES FOR SUMS OF RISKS *

Antonella Campana[1] and Paola Ferretti[2]

[1] Dipartimento di Scienze Economiche, Gestionali e Sociali
Università degli Studi del Molise
(e-mail: campana@unimol.it)

[2] Dipartimento di Matematica Applicata and SSAV
Università Ca' Foscari, Venezia
(e-mail: ferretti@unive.it)

ABSTRACT: In this paper we consider the problem of determining approximations for distortion risk measures of sums of non-independent random variables. First, we give an overview of the recent actuarial literature on distortion risk measures and convex bounds for sums of random variables. Then, we examine the case of discrete risks with identical distribution. Upper and lower bounds for risk measures of sums of risks are presented in the case of concave distortion functions. The result is then extended to cover the case of non necessarily discrete risks. Finally, the attention is devoted to the analysis of the gap between the risk measures of upper and lower bounds, with the aim of minimizing it.

KEYWORDS: Risk measures; dependency of risks; discrete risks with identical distribution; upper and lower bounds; concave risk measures.

1 Introduction

Recently in actuarial literature, the study of the impact of dependence among risks has become a major and flourishing topic: even if in traditional risk theory individual risks have been usually assumed to be independent, this assumption is very convenient for tractability but it is not generally realistic. Think for example to the aggregate claim amount in which any random variable represents the individual claim size of an insurer's risk portfolio. When the risk is represented by residential dwellings exposed to danger of an earthquake in a given location or by adjoining buildings in fire insurance, it is unrealistic to state that individual risks are not correlated, because they are subject to the same claim causing mechanism. Several notions of dependence were introduced in literature to model the fact that larger values of one of the component of a multivariate risk tend to be associated with larger values of the others.

In financial or actuarial situations one often encounters random variables of the type

$$S = \sum_{i=1}^{n} X_i$$

*Partially supported bu MIUR

where the terms X_i are not mutually independent and the multivariate distribution function of the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified but one only knows the marginal distribution functions of the risks. As usual, an insurance risk is defined as a non-negative real-valued random variable X defined on some probability space. We will consider a set Γ of risks with bounded support $[0, c]$.

In such cases, to be able to make decisions, it may be helpful to determine approximations for the distribution of S.

Then, to study sums of random variables of which the marginal distribution is known but the joint distribution is not specified or too complex to work with, it is possible to refer to upper and lower bounds in the sense of convex order, namely to the riskiest portfolio in which the multivariate distribution refers to mutually comonotonic risks, and to the safest portfolio in which the multivariate distribution refers to mutually exclusive risks, respectively.

The present contribution is devoted to the analysis of a particular class of risk measures, namely of distortion risk measures introduced by Wang, 1996. They can be written as

$$W_g(X) = \int_0^{\infty} g(H_X(x))dx \tag{1}$$

where the distortion function g is defined as a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

We are interested in the definition of explicit formulas for distortion risk measures of upper and lower bounds of sums of risks, both in case of discrete identically distributed risks, both in case of identically distributed risks not necessarily discrete.

The framework is that of multivariate risks with the same marginal distributions; in particular, S^c is the sum of the components of $\mathbf{X^c}$, i.e. a random vector with the same marginal distributions of $\mathbf{X}$ and with the comonotonic dependence structure. The context is that of distortion risk measures, that is of measures of risks satisfying

additivity for comonotonic risks

$$W_g(S^c) = \sum_{i=1}^{n} W_g(X_i); \tag{2}$$

positive homogeneity

$$W_g(aX) = a W_g(X) \quad \text{for any non-negative costant } a; \tag{3}$$

translation invariance

$$W_g(X+b) = W_g(X) + b \quad \text{for any costant } b \tag{4}$$

and preservation of first order stochastic dominance. In the particular case of a concave distortion measure, the corresponding distortion risk measure is also sub-additive and it preserves stop-loss order. As it is well-known, examples of concave distortion risk measures are the TVaR and the PH-transform risk measure, whereas the quantile risk measure is not a concave risk measure.

First, we assume that the risks are discrete and identically distributed, then, that they are identically distributed but not necessarily discrete. Starting from the representation of risks as sums of layers, we derive explicit formulas for risk measures of upper and lower bounds of sums of risks, in the particular case of concave distortion risk measures.

In the case of the PH-transform risk measure introduced by Wang (1995), we obtain that the performance of the upper and lower approximations depends on the risk-averse index.

Finally, our attention is devoted to the study of the difference between the risk measures of upper and lower bounds, with the aim of minimizing this gap.

References

DENUIT, M., & LEFEVRE, C. 1997. Some new classes of stochastic order relations among arithmetic random variables, with applications in actuarial sciences. *Insurance: Mathematics and Economics.*, **20**, 197–213.

DHAENE, J., & DENUIT, M. 1999. The safest dependence structure among risks. *Insurance: Mathematics and Economics.*, **25**, 11–21.

DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R., & VYNCKE, D. 2002a. The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics and Economics.*, **31**, 133–161.

DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R., & D., VYNCKE. 2002b. The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics and Economics.*, **31**, 3–33.

DHAENE, J., VANDUFFEL, S., TANG, Q.H., GOOVAERTS, M.J., KAAS, R., & VYNCKE, D. 2004. Solvency capital, risk measures and comonotonicity: a review. *DTEW Research Report 0416, K.U.Leuven*, **0416**, 1–33.

GOOVAERTS, M.J., & DHAENE, J. 1998. On the characterization of Wang's class of premium principles. *Transactions of the 26th International Congress of Actuaries.*, **4**, 121–134.

KAAS, R., DHAENE, J., & GOOVAERTS, M.J. 2000. Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics.*, **27**, 151–168.

WANG, S. 1995. IInsurance pricing and increased limits ratemaking by proportional hazard transforms. *Insurance: Mathematics and Economics.*, **17**, 43–54.

WANG, S. 1996. Premium calculation by transforming the layer premium density. *ASTIN Bulletin.*, **26**, 71–92.

WANG, S., & DHAENE, J. 1998. Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics.*, **22**, 235–242.