

Inference for Stochastic Volatility models: a sequential approach ^(*)

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Abstract: In this paper we propose a sequential Monte Carlo algorithm to estimate a stochastic volatility model with leverage effect. Our idea relies on the auxiliary particle filter method that allows to sequentially evaluate the parameters and the latent processes involved in the dynamic. An empirical application on simulated data is presented to study some empirical properties of the algorithm implemented.

Keywords: Stochastic volatility, auxiliary particle filter, Bayesian estimation.

1. Introduction

Stochastic volatility modeling represents an important topic for financial applications. A general treatment can be found in Ghysels *et al.* (1996). Empirical evidence shows that the use of stochastic volatility leads to an improved fit of the data, both in relation to their descriptive power (Kim *et al.* 1998) as well as for the pricing of derivatives (Bakshi *et al.* 1997 and Pan 2002) with respect to the standard Black and Scholes framework.

The aim of this paper is to propose and implement an efficient sequential procedure in order to estimate a stochastic volatility model with leverage.

The leverage is a really important feature in empirical financial applications and basically describes the relationship between volatility and returns. In general, a negative shock on the returns tends to increase spot volatilities. In the stochastic volatility context this characteristic is described through a non null correlation between the shocks of the equations involved in the model.

The remainder of the paper is organized as follows. The basic model is described in Section 2. Our inferential solution for that class of models is outlined in Section 3. Finally some empirical results based on the study of simulated data are illustrated in section 4.

2. The Model

In this section we introduce the models we consider in our analysis. We analyze a discrete time stochastic volatility model that is an approximation of the popular continuous time diffusion driven by a Brownian motion proposed in Hull and White (1987). In the recent econometric literature this model together with some extensions have been extensively analyzed in Jacquier *et al.* (2004) among others.

Returns y_{t+1} and the unobserved log-volatilities V_{t+1} are described by a non-linear

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state space model as follows

$$y_{t+1} = \exp\{V_t/2\}\epsilon_{t+1} \quad (1)$$

$$V_{t+1} = \mu + \phi V_t + \sigma_\eta \eta_{t+1} \quad (2)$$

in which (ϵ_t, η_t) is a zero mean bivariate Normal random variable with correlation ρ . Furthermore the initial state V_0 is distributed according to $N\left(\frac{\mu}{1-\phi}; \frac{\sigma_\eta^2}{1-\phi^2}\right)$. The parameter ϕ is the persistence of the volatility process, that allows for the volatility clustering, μ is the drift component and σ_η can be interpreted as the volatility of the volatility factor. The correlation ρ is often interpreted as the leverage effect and induces a relationship between volatilities and returns. A negative correlation can be interpreted as the leverage effect, that is, the asymmetric response to positive and negative shocks.

3. Inference

Stochastic volatility models have represented a benchmark for inferential procedures in the last years. We rely our inferential procedure on sequential Monte Carlo algorithms. More precisely we implement a version of the auxiliary particle filter.

For practical applications, it is intuitive to understand that agents need to produce estimates and forecasts in real time. This is the reason why we think sequential methods are appealing from a practical and a theoretical point of view.

The methodology adopted builds on Liu and West (2001). In this framework the goal is to estimate the volatility process together with the parameter vector according to the flow of information available at a given time t , i.e. $y_{1:t} = (y_1, \dots, y_t)$. It is worth noting that the vector θ is time invariant.

The basic idea is to estimate the posterior distribution $p(V_t, \theta | y_{1:t})$. To simplify the notation, θ_t is the posterior conditional on the past up to time t , i.e. $\theta_t \sim p(\theta | y_{1:t})$.

The estimate of the *augmented* state $x_t = (V_t, \theta_t)$ can be based on the auxiliary particle filter of Pitt and Shephard (1999).

Particle filters aim at estimating the latent processes through sequential Monte Carlo simulations. To run the filter, it is required the knowledge of $p(x_0)$, the transition distribution $p(x_{t+1}|x_t)$, $t \geq 0$ and the measurement distribution $p(y_t|x_t)$, $t \geq 1$. The key idea is to approximate the filtering density $p(x_{t+1}|y_{1:t+1})$ by a discrete cloud of points, $\{x_{t+1}^j : j = 1, \dots, N\}$, called particles. This density is estimated as

$$\hat{p}(x_{t+1}|y_{1:t+1}) = \sum_{j=1}^N \omega_{t+1}^j \delta(x_{t+1} - x_{t+1}^j), \quad (3)$$

where ω_{t+1}^j are suitable weights and $\delta(\cdot)$ is an indicator function. Obtaining a sample from (3) is easy if we recur to the importance sampling method in which the proposal $q(x_{t+1}|y_{1:t+1})$ can be set to $p(x_{t+1}|X_t)$ with weights $\omega_{t+1}^j \propto \omega_t^j p(y_{t+1}|x_{t+1}^j)$.

In order to apply the particle filter methodology, we need to *complete* the dynamic of x_t . Liu and West (2001) propose to update θ_t through

$$\theta_{t+1} = \theta_t + \zeta_t, \quad \zeta_t \sim N(0; W_{t+1}). \quad (4)$$

in which W_t is a specified variance matrix.

The resulting algorithm run as follows

0. Simulate N particles for θ_0 from their prior and the volatility from $p(V_0)$
For $t = 1$ to T :

1. Given $x_t^j = (V_t^j, \theta_t^j)$ and ω_t^j , $j = 1, \dots, N$, compute

$$\mu_{t+1}^j = E[V_{t+1}|V_t^j, \theta_t^j]$$

$$m_t^j = a\theta_t^j + (1-a)\bar{\theta}_t$$

2. Draw the integer κ from $\kappa \in \{1, \dots, N\}$ with probabilities

$$g_{t+1}^j \propto \omega_t^j p(y_{t+1}|\mu_{t+1}^j, m_t^j)$$

3. Update θ_{t+1} from $\theta_{t+1}^k \sim N(m_t^k, h^2 \Sigma_t)$

4. Update V_{t+1} from $p(V_{t+1}|V_t^k, \theta_{t+1}^k)$

5. Compute $\omega_{t+1}^k \propto \frac{p(y_{t+1}|V_{t+1}^k, \theta_{t+1}^k)}{p(y_{t+1}|\mu_{t+1}^k, m_t^k)}$

6. Repeat step (2)-(5) N times. Record $x_{t+1}^j = (V_{t+1}^j, \theta_{t+1}^j)$.

The matrix Σ_t and the vector $\bar{\theta}_t$ are suitable estimates of the variance and mean of the posterior distribution whereas the constants h and a are defined in order to avoid *loss of information* troubles that leads to weak approximations of the distributions involved in the filtering procedure⁽¹⁾.

4. Empirical Results

We report some results of the algorithm described above for the model described in (1)-(2). We set the parameters $\theta = (\phi = 0.9, \mu = 0.06, \rho = -0.5, \sigma_\eta = 0.15)$. This setting is consistent with empirical results typical with real data. To initialize the algorithm we define the following priors

$$\phi \sim \text{Beta}(30; 1.5);$$

$$\mu \sim N(0; 10);$$

$$\sigma_\eta \sim \text{IG}(2.5; 0.05);$$

$$(2\rho - 1) \sim \text{Beta}(0.5; 1).$$

The results, based on data set of length 4,000 are reported in Figure 4. The auxiliary particle filter is based on 2,000 particles. For each t the trajectory θ_t is computed as

$$\hat{\theta}_t = \sum_{i=1}^N \omega_t^i \theta_t^i \tag{5}$$

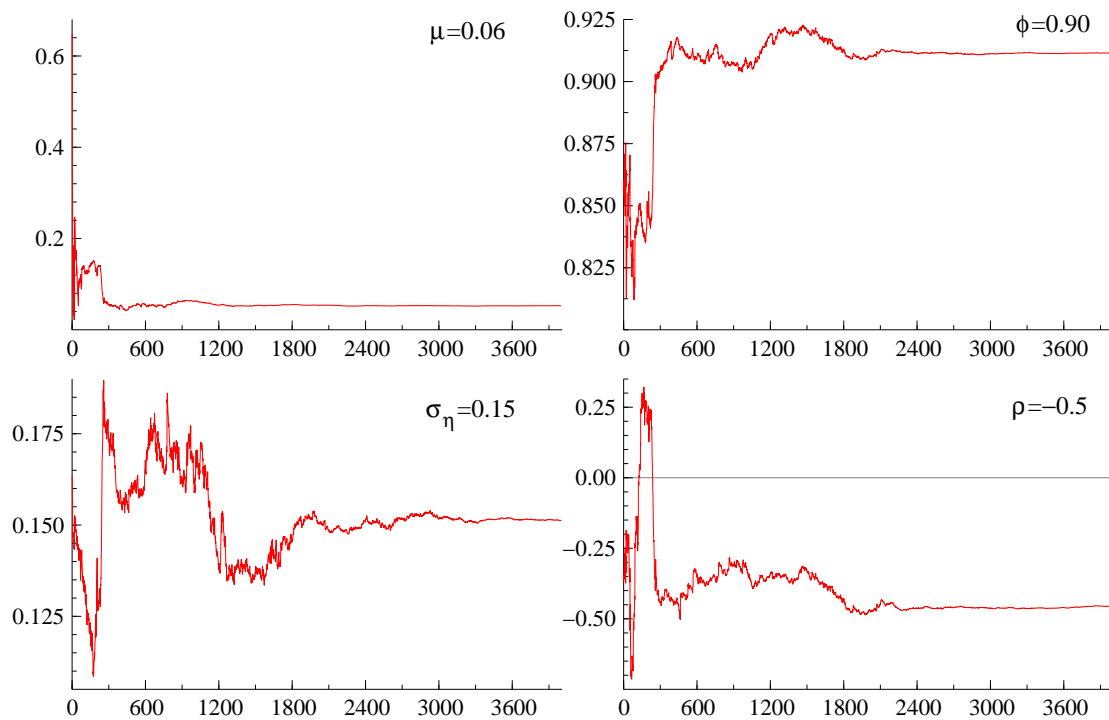
where the weights ω_t^i are the ones that allow to approximate equation 3

It is quite clear that the algorithm produces reliable estimate for the *posterior* mean based on the information available up to time t . Of course this is not a proof for the goodness of this methodology.

However, this sequential procedure seems to give encouraging results and deserves to be studied with more depth. Furthermore, some generalization based on various jump dynamics are currently under investigation.

⁽¹⁾ see Liu and West (2001) for technical details

Figure 1: Time Trajectories for the parameters of interest $\theta = (\phi, \mu, \sigma_\eta, \rho)$.



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