Scaling Laws in Stock Markets. An analysis of prices and volumes

Sergio Bianchi† and Augusto Pianese‡
University of Cassino, Italy
†sbianchi@eco.unicas.it, ‡pianese@unicas.it

Extended abstract

In the last years fractal models have become the focus of many contributions dealing with market dynamics modelization. In this context, one of the key points concerns the estimation of the so called self-similarity parameter, that is the scaling exponent for which the finite-dimensional probability distribution functions (pdf's) relative to different time horizons (time scales) become equal.

More formally, the continuous time, real-valued process $\{X(t),\ t \in T\}$, with $X(0) = 0$, is self-similar with index $H_0 > 0$ (concisely, H_0-ss) if, for any $a \in \mathbb{R}^+$ and any integer k such that $t_1, ..., t_k \in T$, the following equality holds for its finite-dimensional distributions

$$\{X(at_1), X(at_2), ..., X(at_k)\} \stackrel{d}{=} \{a^{H_0}X(t_1), a^{H_0}X(t_2), ..., a^{H_0}X(t_k)\}. \tag{1}$$

A less restrictive definition considers the second-order stationary, real-valued stochastic process $X(t)$, which is said H_0-second order self-similar if − denoted by $Y(t, a)$ its a-lagged increments, namely $Y(t, a) = X(t+a) - X(t)$, and by $\overline{Y}(t, m) = m^{-1} \sum_{\tau=(t-1)m+1}^{tm} Y(\tau, 1)$, $m, t \in \{1, 2, ...\}$ the averaged (over blocks of lenght m) sequence − it holds

$$Var\left(\overline{Y}(t, m)\right) = m^{2(H_0-1)} Var\left(Y(t, 1)\right).$$

Finally, $X(t)$ is said H_0-second order asymptotically self-similar if

$$Var\left(\overline{Y}(t, km)\right) \sim k^{2(H_0-1)} Var\left(\overline{Y}(t, m)\right) \quad \text{as } m \to \infty,\ k \in \{1, 2, ...\}.$$

As equality (1) implies

$$\mathbb{E}(|X(t)|^q) = t^{H_0 q}\mathbb{E}(|X(1)|^q) \tag{2}$$

self-similarity is usually tested via the scaling behaviour of the sample moments of $X(t)$ but this approach has some drawbacks analyzed in Bianchi (2004).

A plethora of contributions have tried to estimate the self-similarity parameter from financial data: Müller et al. (1990) find that the intra-day mean absolute changes of log-prices of four FX spot rates against the U.S. Dollar follow a scaling law, although the distributions strongly differ for different interval sizes. Helms et al. (1984), Cheung and Lai (1993), and Fang et al. (1994) obtain similar results for other data and argue that scaling in finance may be a feature of some spot and futures FX rates and of commodity prices. Estimating the scaling relationships between the volatility of returns at different time intervals of four spot FX rates against U.S. Dollar from 1985 to 1998, Batten and Ellis (1999) find three of them to have scale exponents larger than $\frac{1}{2}$ for all time intervals (ranging from 0.565 to 0.575). Schmitt et al. (2000) find nonlinearity in the log variations of the daily US Dollar/French Franc exchange rate from 1979 to 1993 and argue the multifractal nature of the FX data. Using different techniques on the minute-by-minute FX rates of the Deutsche Mark and of eight Asian currencies in the summer 1997, Karuppiah and Los (2001) find that most FX rates exhibit scaling exponents in the range $0.2 - 0.5$. Gençay and Xu (2003) find that the 5-min returns of the US Dollar versus the Deutsche Mark series show different slopes for these powers and argue that the nonlinearity of the scaling exponent indicates multifractality of returns. Soofi and Galka (2003) use the theory of nonlinear dynamical systems to measure the complexity of the exchange rates of British Pound and Japanese Yen versus US Dollar and find discernible nonlinear structure in the returns of the Dollar/Pound daily rate for the period 1973-1989.

In Bianchi (2004) the definition (1) is equivalently reformulated introducing a proper metric on the space of the rescaled pdf's as follows. Let $\mathcal{A}$ be any bounded subset of $\mathbb{R}^+$, $\mathfrak{a} = \min(\mathcal{A})$ and $\mathfrak{A} = \max(\mathcal{A}) < \infty$, for any $a \in \mathcal{A}$, consider the k-dimensional distribution Φ of the a-lagged process $X(at)$. Equality (1) becomes

$$\Phi_{\mathbb{X}(a)}(\mathbf{x}) = \Phi_{a^{H_0}\mathbb{X}(1)}(\mathbf{x}) \tag{3}$$

where $\mathbb{X}(a) = (X(at_1), ..., X(at_k))$ and $\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k$. It follows

$$\begin{aligned} \Phi_{a^{-H}\mathbb{X}(a)}(\mathbf{x}) &= \Pr\left(a^{-H}X(at_1) < x_1, ..., a^{-H}X(at_k) < x_k\right) = \text{by } H_0\text{-ss} = \\ &= \Pr\left(a^{H_0-H}X(t_1) < x_1, ..., a^{H_0-H}X(t_k) < x_k\right) = \Phi_{a^{H_0-H}\mathbb{X}(1)}(\mathbf{x}) = \\ &= \Pr\left(X(t_1) < a^{H-H_0}x_1, ..., X(t_k) < a^{H-H_0}x_k\right) = \Phi_{\mathbb{X}(1)}(a^{H-H_0}\mathbf{x}) \end{aligned} \tag{4}$$

So, denoted by ρ the distance function induced by the sup-norm $\|\cdot\|_\infty$ on the space Ψ_H of the (absolutely continuous) k-dimensional pdf's of $\{a^{-H}X(at)\}$ with respect to the set $\mathcal{A}$, the diameter of the metric space (Ψ_H, ρ)

$$\delta^k(\Psi_H) = \sup_{\mathbf{x}\in\mathbb{R}^k} \sup_{a_i,a_j\in\mathcal{A}} \left|\Phi_{a_i^{-H}\mathbb{X}(a_i)}(\mathbf{x}) - \Phi_{a_j^{-H}\mathbb{X}(a_j)}(\mathbf{x})\right| \tag{5}$$

measures the *discrepancy* among the rescaled distributions. Due to trivial properties of the supremum with respect to the sets in $\mathcal{A}$, once the parameter H has been fixed, relation (5) becomes the statistics of the well-known Kolmogorov-Smirnov goodness of fit test and this enriches the self-similarity analysis with an inferential support.

In this work we observe that, given two any lags a and b, for a true self-similar process it follows from (3) that

$$\Phi_{a^{-H_0}\mathbb{X}(a)}(\mathbf{x}) = \Phi_{b^{-H_0}\mathbb{X}(b)}(\mathbf{x})$$

and, more generally, that (5) can be written as

$$\delta^k(\Psi_H) = \sup_{\mathbf{x}\in\mathbb{R}^k} \left|\Phi_{b^{-H}\mathbb{X}(b)}(\mathbf{x}) - \Phi_{a^{-H}\mathbb{X}(a)}(\mathbf{x})\right|. \tag{6}$$

The last relation means that the scaling properties of the process can be analyzed by pairwise comparisons of the time horizons, so relaxing the constraint to refer all the scales to the sole unit lag horizon.

Some technicalities are discussed in the paper and a wide-range analysis is performed on several stock indexes both for the price increments and the volumes. In particular, the number of the traded shares has been considered to check the idea founding the multifractal market model introduced by Calvet and Fisher (2002). This model suggests to replace the physical time by the so called 'trading time' in order to take into account the (variable) number of shares traded per unit time interval.

Our analysis generates in the three-dimensional space (a, b, H_0) scaling surfaces whose points - when self-similarity cannot be rejected - generally differ with each pair of time horizons and shows how taking into consideration just one base time horizon could lead to wrong conclusions about the process' scaling law.

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