OPTIMAL RISK SHARING WITH NON-MONOTONE MONETARY FUNCTIONS

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ABSTRACT: Recently, Jouini et al. (2005) studied the problem of optimal sharing of aggregate risks between two economic agents endowed with monetary utility functions. Strongly inspired by this paper, we consider the analogous problem when admitting any number of agents characterized by non-necessarily monotone choice criterions, and generalize some of the existence and characterization results given there. Moreover, the introduction of the *best* monotone approximation of non-monotone functionals (as suggested in Maccheroni *et al.* , 2005) allows us to make a comparison between our original problem and the one which only involves *ad hoc* monotone functionals. The explicit calculation of optimal risk sharing rules is provided for particular cases, when agents are endowed with well known preference relations.

KEYWORDS: risk measures, risk sharing, convex duality

1 Formulation of the Problem

We work in a simple model consisting of two dates: today, where everything is known, and a fixed future date, say tomorrow, where we suppose that a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given. We admit, as possible financial positions occurring tomorrow, all the essentially bounded random variables, and we evaluate them by means of functionals defined on $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and fulfilling suitable properties. The approach we adopt to study problems of optimal risk exchange is the same as Barrieu & El Karoui (2005) and Jouini et al. (2005), where agents' preferences are represented via monetary utility functions, that, up to the sign, are convex risk measures in the sense of Föllmer & Schied (2004) (for this axiomatic approach see the pioneering work of Artzner *et al.* , 1999). We consider an aggregate of n agents, for some $n \in \mathbb{N}$, endowed with initial risky positions $(\xi)_{i=1}^n \in L^\infty$, and characterized by choice functionals $U_i : L^\infty \to \mathbb{R}$ satisfying the following properties for any $i = 1, \ldots, n$:

- concavity: $U_i(\alpha X + (1-\alpha)Y) \geq \alpha U_i(X) + (1-\alpha)U_i(Y), \forall X, Y \in L^\infty, \alpha \in (0,1)$;
- translation-invariance: $U_i(X + c) = U_i(X) + c, \forall X \in L^\infty, c \in \mathbb{R}$;
- continuity w.r.to the supremum norm on L^∞.

Moreover, by possibly adding a constant, we can assume the normalization property: $U_i(0) = 0$, for all $i = 1, \ldots, n$.

At this point, the question that arises is whether the agents may re-share the total risk $X = \sum_{i=1}^n \xi_i$ in order to make their situation better, where 'better' has the meaning of 'more satisfactory' in the sense of the choice criterions $(U_i)_{i=1}^n$. Note that, since we do not require monotonicity on the preference functionals, we include in our study some widely used non-monotone criterions, such as the mean-variance (resp. the standard-deviation) principle:

$$U(X) = \mathbf{E}[X] - \delta f(X), \text{ with } \delta > 0,\ f(X) = Var(X)\ (\text{resp.} (Var(X))^{1/2}). \tag{1}$$

For criterions satisfying these properties we generalize some representation results which are well known for the class of monetary utility functions, such as the dual representability in L^1 under the Fatou property, and the quantile representation under the law-invariance property. In this more general setting, we can also extend some existence and characterization results given in Jouini et al. (2005) for the optimal solutions to the risk sharing problem (see Theorems A and B below), that in our case can be written as follows:

$$\begin{cases} \sup\limits_{\sum_i X_i = X} \sum\limits_{i=1}^n U_i(X_i), \quad X = \sum\limits_{i=1}^n \xi_i \\ (IR) \quad U_i(X_i) \geq U_i(\xi_i),\ \forall i = 1,..,n. \end{cases} \tag{2}$$

It consists in the *sup-convolution* of the agents' choice functionals, under side constraints depending on their initial risk endowments.

Let us recall the notion of Pareto optimality:

Definition An n-tuple $(X_i)_{i=1}^n \in L^\infty$ with $\sum_i X_i = X$ is said a Pareto Optimal Allocation (POA) of the total risk X if for any $(Y_i)_{i=1}^n \in L^\infty$ with $\sum_i Y_i = X$ and $U_i(Y_i) \geq U_i(X_i)$ for all i, then $U_i(Y_i) = U_i(X_i)$ for all i.

Now the property of translation-invariance allows us to solve problem (2) in two separate steps. First, we consider the unconstrained optimization problem

$$U(X) := U_1 \Box \cdots \Box U_n(X) = \sup \left\{ \sum_{i=1}^n U_i(X_i) : X_i \in L^\infty,\ \sum_{i=1}^n X_i = X \right\}, \tag{3}$$

which defines the shape of the optimal contract and produces the Pareto optimal allocations of the total risk (see Theorem A-(i) below). Afterwards, we impose the *individual rationality* (IR) constraints, that characterize the set of suitable prices of the optimal contract and select, among all the POAs, those that make each agent willing to enter into the transaction (see Theorem A-(ii) below).

2 Main Results

Here we state the characterization and existence results announced before (see Jouini et al. (2005) for the case of two agents endowed with monetary utility functions).

Theorem A [Characterization] Let $(U_i)_{i=1}^n$ be preference functionals satisfying our assumptions and such that the functional U defined in (3) is proper. Then, for a given allocation $(X_1, \dots, X_n)$ of the total risk X, the following statements hold:

(i) $(X_i)_{i=1}^n$ is Pareto optimal if and only if $(X_i)_{i=1}^n$ solves problem (3);
(ii) if $(X_i)_{i=1}^n$ is a POA and $(\pi_i)_{i=1}^n \in \mathbb{R}$ is such that $\sum_{i=1}^n \pi_i = 0$, then the allocation $(X_i - \pi_i)_{i=1}^n$ solves problem (2) if and only if $\pi_i \leq U_i(X_i) - U_i(\xi_i)$ for all i.

Recall that a functional U is said law-invariant if $U(X)$ depends only on the law of X.

Theorem B [Existence] Let $(U_i)_{i=1}^n$ be law-invariant choice functionals satisfying our assumptions. Then, for any $X \in L^\infty$, the set of POAs of X is non-empty.

At this point, the introduction of the best monotone approximation of non-monotone functionals (as given in Maccheroni *et al.* , 2005) allows us to compare the behaviour of monotone and non-monotone agents when facing the risk sharing problem. In particular, we may associate to our original problem a new one which only involves ad hoc monotone functionals, and show that there is a strict link between their solutions.

In this purpose, for any preference functional U_i we define the minimal monotone functional U_i^m that dominates U_i:

$$U_i^m(X) := \sup\{U_i(Y) : Y \in L^\infty \text{ and } Y \leq X\}, \; \forall X \in L^\infty,$$

and by means of these new criterions we formulate the following sup-convolution problem

$$\widetilde{U}(X) := U_1^m \square \cdots \square U_n^m(X). \tag{4}$$

Theorem C [Comparison] Let $(U_i)_{i=1}^n$ be preference functionals satisfying our assumptions, such that at least one of these is monotone and the functional U defined in (3) is proper. Then, for any aggregate risk $X \in L^\infty$, any solution to problem (3) solves problem (4) as well. In particular we have that $U(X) = \widetilde{U}(X)$ for all $X \in L^\infty$.

We especially obtain interesting results when the non-monotone agents have preferences of mean-variance type. In this case we find that the optimal redistribution of the total risk is not sensitive to the lack of monotonicity by some agents, that is, problem (3) and problem (4) admit the same set of solutions.

3 Explicit Characterization of Optimal Risk Sharing Rules

Besides these general results, we study particular risk exchange problems where the involved agents are endowed with well-known choice criterions and, thanks to the powerful tool of convex duality theory, we provide the explicit calculation of the optimal solutions.

Let us recall that the Value at Risk is the opposite of the upper-quantile function: $V@R_t(X) = -\inf\{x \in \mathbb{R} : F_X(x) > t\}$ for any $t \in [0,1)$ ($V@R_1(X) = -\text{ess sup}X$), and that, from it, the Average Value at Risk $AV@R_\lambda$ is defined as follows:

$$-AV@R_\lambda(X) := -\frac{1}{\lambda}\int_0^\lambda V@R_t(X)dt, \quad \lambda \in (0,1]. \tag{5}$$

Definition A functional U on L^∞ is said strictly risk-averse conditionally on any event if it satisfies the following property:

(S) $U(X) < U(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A)$ for any $A \in \mathcal{F}$ and $X \in L^\infty$ such that $\mathbb{P}(A) > 0$ and $\text{essinf}_A X < \text{esssup}_A X$.

For instance, the mean-variance and the standard-deviation principles given in (1) satisfy this property, and therefore they are included in the following example.

• **AV@R-Agent vs Agents with Property (S)**
Let U_1 be the $AV@R_\lambda$-criterion given in (5), and let U_2 be a law-invariant functional satisfying our assumptions and property (S). Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to a constant) POA of X, given by

$$(X_1, X_2) := (-(X-l)^- + (X-u)^+, (l \vee X) \wedge u), \quad \text{for some } l, u \in \mathbb{R}.$$

Besides the $AV@R$-criterion, a monotone functional which is widely used to shape agents' preferences is the entropic utility (with risk tolerance coefficient $\gamma > 0$):

$$U_\gamma(X) := -\gamma \ln \mathbf{E}\big[\exp\big(-X/\gamma\big)\big]. \tag{6}$$

• **AV@R vs Entropic vs Mean-variance vs Standard-Deviation**
Let U_0 be the $AV@R_\lambda$-criterion (5), U_1 the entropic utility (6), U_2 the mean-variance principle with $\delta_1 > 0$, and U_3 the standard-deviation principle with $\delta_2 > 0$. Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to constants summing up to zero) POA $(X_i)_{i=0}^3$ of X, such that $X_0 = -(X-k)^-$, for some $k \in \mathbb{R}$, whereas X_1 (resp. X_2, X_3) is a convex (resp. concave) function of $(X \vee k)$, with X_2 proportional to X_3.

What stems from the cases studied here is that the optimal redistribution of the risk often leads to simple contracts consisting in the exchange of European options written on the total risk or in a proportional sharing of it. In this way we get typical forms of insurance contracts, such as stop-loss and quota-share rules. These examples also reveal peculiar attitudes linked to the respective preference relations: the conservative behaviour of the entropic-agent, as well as the inclination of the $AV@R$-agent towards taking extreme risks.

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